Characterisations of Classical and Non-classical states of Quantised Radiation

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(February 1, 2008)

03.65.Fd, 42.50.Dv, 42.50.Lc

A new operator based condition for distinguishing classical from non-classical states of quantised radiation is developed. It exploits the fact that the normal ordering rule of correspondence to go from classical to quantum dynamical variables does not in general maintain positivity. It is shown that the approach naturally leads to distinguishing several layers of increasing nonclassicality, with more layers as the number of modes increases. A generalisation of the notion of subpoissonian statistics for two-mode radiation fields is achieved by analysing completely all correlations and fluctuations in quadratic combinations of mode annihilation and creation operators conserving the total photon number. This generalisation is nontrivial and intrinsically two-mode as it goes beyond all possible single mode projections of the two-mode field. The nonclassicality of pair coherent states, squeezed vacuum and squeezed thermal states is analysed and contrasted with one another, comparing the generalised subpoissonian statistics with extant signatures of nonclassical behaviour.

I. INTRODUCTION

Electromagnetic radiation is intrinsically quantum mechanical in nature. Nevertheless it has been found extremely fruitful, at both conceptual and practical levels, to designate certain states of quantised radiation as being essentially "classical", and others as being "non-classical" [1]. It is the latter that show the specific quantum features of radiation most sharply. Some of the well known signs of nonclassicality in this context are quadrature squeezing [2], antibunching [3] and subpoissonian photon statistics [4].

The purposes of this paper are to present a new physically equivalent way of distinguishing classical from nonclassical states of radiation, dual to the customary definition and based on operator properties; to point out the existence of several levels of classical behaviour, with a structure that gets progressively more elaborate as the number of modes increases; and finally to give a complete discussion of signatures of nonclassical photon statistics for two-mode fields, working at the level of fluctuations in photon numbers.

The contents of this paper are arranged as follows In section II we develop a criterion based on operator expectation values, to distinguish between classical and nonclassical states of radiation. The basic idea is that the normal ordering rule of correspondence between classical dynamical variables and quantum operators, while being linear and translating reality into hermiticity, does not respect positivity. If this potential nonpositivity does not show up in the expectation values of operators in a certain state, then that state is classical; otherwise it is nonclassical. Section III explores this new approach further and shows that, as the number of independent modes increases, the classification of quantum states gets progressively finer; several levels of nonclassicality emerge. This is shown in detail for one and two mode fields, and then the trend becomes clear. Section IV analyses in complete detail the properties of two mode photon number fluctuations, stressing the freedom to choose any normalised linear combination of the originally given modes as a variable single mode. The well known Mandel parameter criterion [5] for sub-Poissonian statistics for a single mode field is extended in full generality to a matrix inequality in the two-mode case. It is shown that certain consequences of this inequality transcend the set of all single mode projections of it, and are thus intrinsically two-mode in character. Explicit physically interesting examples of this situation are provided, and the well known pair coherent states are also examined from this point of view. Section V presents some concluding remarks.

II. THE DISTINCTION BETWEEN CLASSICAL AND NON-CLASSICAL STATES - AN OPERATOR CRITERION

We deal for simplicity with states of a single mode radiation field, though our arguments generalise immediately to any number of modes. The photon creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ obey the customary commutation relation

$$[\hat{a},\ \hat{a}^\dagger] \equiv \hat{a}\ \hat{a}^\dagger - \hat{a}^\dagger\ \hat{a} = 1\cdot \tag{2.1}$$

The coherent states $|z\rangle$ are right eigenstates of $\hat{a}$ with (a generally complex) eigenvalue z; they are related to the states $|n\rangle$ of definite photon number (eigenstates of $\hat{a}^\dagger\ \hat{a}$) in the standard way:

$$\begin{aligned}
|z\rangle &= e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle\ ,\\
\hat{a}|z\rangle &= z|z\rangle\ ,\\
\langle z'|z\rangle &= \exp\left(-\frac{1}{2}|z'|^2-\frac{1}{2}|z|^2+z'^*z\right)\ ;\\
|n\rangle &= \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle\ ,\\
\hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle\ ,\\
\langle n'|n\rangle &= \delta_{n'n}\cdot
\end{aligned} \tag{2.2}$$

A general (pure or mixed) state of the one-mode field is described by a corresponding normalised density matrix $\hat{\rho}$:

$$\hat{\rho}^\dagger = \hat{\rho} \geq 0, \quad \mathrm{Tr}\ \hat{\rho} = 1\cdot \tag{2.3}$$

It can be expanded in the so-called diagonal coherent state representation [6]:

$$\begin{aligned}
\hat{\rho} &= \int \frac{d^2z}{\pi}\ \phi(z)\ |z\rangle\langle z|\ ,\\
&\int \frac{d^2z}{\pi}\ \phi(z) = 1\cdot
\end{aligned} \tag{2.4}$$

While hermiticity of $\hat{\rho}$ corresponds to reality of the weight function $\phi(z)$, the latter is in general a singular mathematical quantity, namely a distribution of a well-defined class.

The conventional designation of $\hat{\rho}$ as being classical or nonclassical is based on the properties of $\phi(z)$. Namely, $\hat{\rho}$ is said to be classical if $\phi(z)$ is everywhere non-negative and not more singular than a delta function [1]:

$$\begin{aligned}
\hat{\rho}\ \text{classical} &\Leftrightarrow \phi(z) \geq 0,\ \text{no worse than delta function}\ ,\\
\hat{\rho}\ \text{nonclassical} &\Leftrightarrow \phi(z) \not\geq 0\cdot
\end{aligned} \tag{2.5}$$

It is clear that the conditions to be classical involve an infinite number of independent inequalities, since $\phi(z) \geq 0$ has to be obeyed at each point z in the complex plane. This is true despite the fact that the condition $\hat{\rho} \geq 0$ means that the "values" of $\phi(z)$ at different points z are not quite "independent". One realises this by recalling that every classical probability distribution over the complex plane is certainly a possible choice for $\phi(z)$, with the corresponding $\hat{\rho}$ being classical.

The above familiar definition of classical states deals directly with $\hat{\rho}$ and $\phi(z)$. Now we develop a dual, but equivalent, definition based on operators and their expectation values. As is well known, the representation (2.4) for $\hat{\rho}$ is closely allied to the normal ordering rule for passing from classical c-number dynamical variables to quantum operators. Within quantum mechanics we know that an operator $\widehat{F}$ is completely and uniquely determined by its diagonal coherent state matrix elements (expectation values) $\langle z|\widehat{F}|z\rangle$. Moreover, hermiticity of $\widehat{F}$ and reality of $\langle z|\widehat{F}|z\rangle$ are precisely equivalent. Any (real) classical function $f(z^*, z)$ determines uniquely, by the normal ordering rule of placing $\hat{a}^\dagger$ always to the left of $\hat{a}$ after substituting $z \to \hat{a}$ and $z^* \to \hat{a}^\dagger$, a corresponding (hermitian) operator $\widehat{F}_N$ as follows:

<u>Normal ordering rule</u>

$$
\begin{aligned}
f(z^*,z) &\longrightarrow \widehat{F}_N \ , \\
\langle z|\widehat{F}_N|z\rangle &= f(z^*,z)\ , \\
f \text{ real } &\Leftrightarrow \widehat{F}_N \text{ hermitian}
\end{aligned}
\tag{2.6}
$$

The connection with the representation (2.4) for $\hat{\rho}$ is given by

$$
\text{Tr}\,\left(\hat{\rho}\,\widehat{F}_N\right) = \int \frac{d^2z}{\pi}\,\phi(z)\,f(z^*,z)\cdot
\tag{2.7}
$$

It is an important property of the normal ordering rule that, while it translates classical reality to quantum hermiticity, *it does not preserve positive semidefiniteness.* More explicitly, while by eq. (2.6) $\widehat{F}_N \geq 0$ implies $f(z^*,z)\geq 0$, *the converse is not true.* Here are some simple examples of nonnegative classical real $f(z^*,z)$ leading to indefinite hermitian $\widehat{F}_N$:

$$
\begin{aligned}
&f(z^*,z) = (z^*+z)^2 \longrightarrow \widehat{F}_N = \left(\hat{a}^\dagger+\hat{a}\right)^2 - 1\ ; \\
&f(z^*,z) = (z^*+z)^4 \longrightarrow \widehat{F}_N = \left(\left(\hat{a}^\dagger+\hat{a}\right)^2 - 3\right)^2 - 6\ ; \\
&f(z^*,z) = e^{-z^*z}\sum_{n=0}^{\infty}\frac{C_n}{n!}\,z^{*n}z^n \rightarrow \widehat{F}_N = \sum_{n=0}^{\infty} C_n|n\rangle\langle n|\cdot
\end{aligned}
\tag{2.8}
$$

In the last example, the real constants C_n can certainly be chosen so that some of them are negative while maintaining $f(z^*,z)\geq 0$; this results in $\widehat{F}_N$ being indefinite.

We thus see that when the normal ordering rule is used, every $\widehat{F}_N \geq 0$ arises from a unique $f(z^*,z)\geq 0$, but some (real) $f(z^*,z)\geq 0$ lead to (hermitian) indefinite $\widehat{F}_N$. So in a given quantum state $\hat{\rho}$, the operator $\widehat{F}_N$ corresponding to a nonnegative real classical $f(z^*,z)$ could well have a negative expectation value. *If this never happens, then $\hat{\rho}$ is classical.* That is, as we see upon combining eqns.(2.5) (2.7): if for every $f(z^*,z)\geq 0$ the corresponding $\widehat{F}_N$ has a nonnegative expectation value even though $\widehat{F}_N$ may be indefinite, then $\hat{\rho}$ is classical. Conversely, $\hat{\rho}$ is nonclassical if there is at least one $f(z^*,z)\geq 0$ which leads to an indefinite $\widehat{F}_N$ whose expectation value is negative.

We can convey the content of this dual operator way of defining classical states also as follows: while the normal ordering rule allows for the appearance of "negativity" in an operator $\widehat{F}_N$ even when none is present in the corresponding classical $f(z^*,z)$, in a classical state such negativity never shows up in expectation values.

Purely by way of contrast, we compare the above with what obtains when the antinormal ordering rule - substituting $z\rightarrow\hat{a},\ z^*\rightarrow\hat{a}^\dagger$ followed by placing $\hat{a}$ to the left, $\hat{a}^\dagger$ to the right - is used to pass from classical $f(z^*,z)$ to quantum $\widehat{F}$ [7]. In place of eqns.(2.6) (2.7) we have:

<u>Antinormal ordering rule</u>

$$
\begin{aligned}
\text{real } f(z^*,z) \rightarrow \text{hermitian } \widehat{F}_A &= \int \frac{d^2z}{\pi}\,f(z^*,z)\,|z\rangle\langle z|\ , \\
\text{Tr}\,\left(\hat{\rho}\,\widehat{F}_A\right) &= \int \frac{d^2z}{\pi}\,\langle z|\hat{\rho}|z\rangle\,f(z^*,z)\cdot
\end{aligned}
\tag{2.9}
$$

Now the situation is that $f(z^*,z)\geq 0$ certainly implies $\widehat{F}_A\geq 0$, but some $\widehat{F}_A\geq 0$ arise from indefinite classical $f(z^*,z)$. A simple example is:

$$
f(z^*,z) = (z^*+z)^2 - 1 \longrightarrow \widehat{F}_A = \left(\hat{a}^\dagger+\hat{a}\right)^2\cdot
\tag{2.10}
$$

Thus the function and operator relationships are opposite to what we found in the normal ordering case. The direct characterization of classical states (whose definition in any case is a convention based on the normal ordering rule) is however not very convenient with the antinormal ordering convention.

In passing we may note an interesting aspect of the Weyl or symmetric rule of ordering [8], and the associated Wigner distribution description of a state $\hat{\rho}$ [9]. Here it is more convenient to deal with the real and imaginary parts of z and $\hat{a}$:

$$
\begin{aligned}
&z = \frac{1}{\sqrt{2}}\,(q+i\,p), \quad z^* = \frac{1}{\sqrt{2}}(q-i\,p)\ ; \\
&\hat{a} = \frac{1}{\sqrt{2}}\,(\hat{q}+i\,\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q}-i\,\hat{p})\ , \\
&[\hat{q},\ \hat{p}] = i\ \cdot
\end{aligned}
\tag{2.11}
$$

The Weyl ordering rule maps single classical exponentials into single operator exponentials,

$$\exp(i\,\lambda\,q + i\,\mu\,p) \rightarrow \exp(i\,\lambda\,\hat{q} + i\,\mu\,\hat{p})\;, \quad \lambda \text{ and } \mu \text{ real}\;; \tag{2.12}$$

and then extends this by linearity and Fourier transformation to general functions:
Weyl rule

$$\begin{aligned} f(q,p) &= \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu\; \tilde{f}(\lambda,\mu)\exp(i\,\lambda\,q + i\,\mu\,p) \longrightarrow \\ \widehat{F}_W &= \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu\; \tilde{f}(\lambda,\mu)\exp(i\,\lambda\,\hat{q} + i\,\mu\,\hat{p})\;, \\ & f \text{ real} \Longleftrightarrow \widehat{F}_W \text{ hermitian} \end{aligned} \tag{2.13}$$

The Wigner distribution $W(q,p)$ for a state $\hat{\rho}$ is given in terms of the configuration space matrix elements of $\hat{\rho}$:

$$W(q,p) = \int_{-\infty}^{\infty} dq' \langle q - \frac{1}{2}q'|\hat{\rho}|q + \frac{1}{2}q'\rangle\; e^{ipq'}\;, \tag{2.14}$$

and the rule for expectation values ties together eqns. (2.13) (2.14):

$$\mathrm{Tr}\,\left(\hat{\rho}\;\widehat{F}_W\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\;dp\;W(q,p)\;f(q,p)\,\cdot \tag{2.15}$$

From the algebraic point of view the Weyl rule stands "midway" or symmetrically between the normal and the antinormal ordering rules. This may lead one to hope that it removes the mismatch and resolves the problem of preserving positivity in both directions in passing between $f(q,p)$ and $\widehat{F}_W$. However this does not happen at all. As is well known there are nonnegative $f(q,p)$ yielding indefinite $\widehat{F}_W$, and nonnegative $\widehat{F}_W$ leading back to indefinite $f(q,p)$. Here are simple examples:

$$\begin{aligned} f(q,p) = \delta(q)\delta(p) &\longrightarrow \widehat{F}_W = \text{parity operator with eigenvalues } \pm 1\;; \\ \widehat{F}_W = |1\rangle\langle 1| &\longrightarrow f(q,p) = \frac{2}{\pi}\left(q^2 + p^2 - \frac{1}{2}\right)\exp\left(-q^2 - p^2\right)\cdot \end{aligned} \tag{2.16}$$

While these remarks illuminate in terms of operator properties the relations among the three ordering rules, the classification of states $\hat{\rho}$ into classical and nonclassical ones is based most simply on the normal ordering rule. It is clear that all these considerations extend easily to any number of modes of radiation.

While our operator based approach to the identification of nonclassical states is conceptually complete, we note that most of the extant criteria of nonclassicality involve a simple extension of our formalism. Namely, one often has to consider nonlinear functions of expectation values of several operators, which cannot be simply expressed as the expectation value of a single state-independent or autonomous operator. This is always so when one deals with fluctuations. Thus quadrature squeezed non-classical states are defined via inequalities involving the fluctuations $(\Delta\hat{q})^2$ and $(\Delta\hat{p})^2$. Similarly amplitude squeezing or sub-Poissonian statistics deals with the fluctuation in photon number. These remarks apply also to other criteria of non-classicality such as higher order squeezing [10] and the one based on matrices constructed out of factorial moments of the photon number distribution [11]. The one interesting exception to these remarks and which is fully covered by our formalism, is the case of antibunching [3]. Here one is concerned with the expectation values of the single time-dependent operator

$$\hat{F}(t, t+\tau) =: \hat{I}(t+\tau)\hat{I}(t) : - : \hat{I}(t)\hat{I}(t) : \tag{2.17}$$

where $\hat{I}(t)$ is the intensity operator at time t. In this sense, the criterion for antibunching is qualitatively different from the other familiar ones.

III. LEVELS OF CLASSICALITY

A. The single mode case

We begin again with the single mode situation and hereafter deal exclusively with the normal ordering prescription. (Therefore the subscript N on $\widehat{F}_N$ will be omitted). Suppose we limit ourselves to classical functions $f(z^*, z)$ which are real, nonnegative and phase invariant, that is, invariant under $z \rightarrow e^{i\alpha} z$. An independent and complete set of these can be taken to be

$$f_n(z^*, z) = e^{-z^* z}\, z^{*n} z^n / n!\ , \quad n = 0,\ 1,\ 2,\ \ldots, \tag{3.1}$$

since they map conveniently to the number state projection operators:

$$f_n(z^*, z) \longrightarrow \widehat{F}^{(n)} = |n\rangle\langle n|\ , \quad n = 0,\ 1, 2,\ \ldots \tag{3.2}$$

A general real linear combination $f(z^*, z) = \sum\limits_n C_n\ f_n(z^*, z)$, even if nonnegative, may lead to an indefinite $\widehat{F}$, as seen at eqn. (2.8).

If we are interested only in the expectation values of such variables, we are concerned only with the probabilities $p(n)$ for finding various numbers of photons; for this purpose an angular average of $\phi(z)$ is all that is required:

$$\begin{aligned} p(n) &= \langle n|\hat{\rho}|n\rangle = \mathrm{Tr}\ (\hat{\rho}|n\rangle\langle n|) \\ &= \int \frac{d^2 z}{\pi}\ \phi(z)\ e^{-z^* z} z^{*n} z^n / n! \\ &= \int_0^\infty dI\ P(I)\ e^{-I}\ I^n / n!\ , \\ P(I) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta\ \phi\left(I^{1/2}\ e^{i\theta}\right) \cdot \end{aligned} \tag{3.3}$$

Now while $\phi(z) \geq 0$ certainly implies $P(I) \geq 0$, the converse is not true. Thus one is led to a three-fold classification of quantum states $\hat{\rho}$ [12]:

$$\begin{aligned} \hat{\rho}\ \text{classical} &\Longleftrightarrow \phi(z) \geq 0\ ,\ \text{hence}\ P(I) \geq 0\ ; \\ \hat{\rho}\ \text{semiclassical} &\Longleftrightarrow P(I) \geq 0\ \text{but}\ \phi(z) \ngeq 0\ ; \\ \hat{\rho}\ \text{strongly nonclassical} &\Longleftrightarrow P(I) \ngeq 0\ ,\ \text{so necessarily}\ \phi(z) \ngeq 0 \cdot \end{aligned} \tag{3.4}$$

The previous definition (2.5) of nonclassical $\hat{\rho}$ based on $\phi(z)$ alone is now refined to yield two subsets of states, the semiclassical and the strongly nonclassical. The semiclassical states do have the following property:

$$\hat{\rho}\ \text{semiclassical} \Longrightarrow\ \mathrm{Tr}\left(\hat{\rho}\widehat{F}\right) \geq 0\ \text{if}\ f(z^*, z) = \sum_{n=0}^{\infty} C_n\ f_n(z^*, z) \geq 0 \cdot \tag{3.5}$$

However, in addition, there would definitely be some phase noninvariant $f(z^*, z) \geq 0$ for which $\widehat{F}$ is indefinite and $\mathrm{Tr}\left(\hat{\rho}\widehat{F}\right) < 0$. It is just that this extent of nonclassicality in $\hat{\rho}$ is not revealed by the expectation values of phase invariant variables, or at the level of the probabilities $p(n)$ [13].

It is clear that the classification (3.4) is $U(1)$ or phase invariant. That is, $\hat{\rho}$ retains its classical, semiclassical or strongly non-classical character under the transformation $\phi(z) \rightarrow \phi'(z) = \phi(z e^{i\alpha})$

As examples of interesting inequalities obeyed if $\hat{\rho}$ is either classical or semiclassical, we may quote the following involving the factorial moments of the photon number probabilities $p(n)$:

$$\begin{aligned} \gamma_m &= \mathrm{Tr}\left(\hat{\rho}\hat{a}^{\dagger m}\hat{a}^m\right) \\ &= \int \frac{d^2 z}{\pi}\ \phi(z)\ (z^* z)^m \end{aligned}$$

$$= \int_0^\infty dI\, P(I)\, I^m$$

$$= \sum_{n=m}^\infty p(n)\, n!/(n-m)! \ \geq 0, \quad m = 0,\ 1,\ 2, \ldots;$$

$$\hat{\rho} \text{ classical or semiclassical} \Longleftrightarrow P(I) \geq 0 \Longrightarrow \gamma_m\, \gamma_n \leq \gamma_{m+n} \leq \sqrt{\gamma_{2m}\gamma_{2n}} \tag{3.6}$$

Violation of any one of these inequalities implies $\hat{\rho}$ is strongly nonclassical.

The inequalities quoted in eq. (3.6) above clearly involve an infinite subset of the photon number probabilities $p(n)$. However one can easily construct far simpler inequalities involving a small number of the $p(n)$'s, violation of any of which also implies that $\hat{\rho}$ is strongly non-classical. For example, from eqs. (3.1) (3.2), for any nonnegative integer n_0 and any real a, b we have the correspondence

$$f(z^\star, z) = e^{-z^\star z}\, \frac{(z^\star z)^{n_0}}{n_0!}\, (a + bzz^\star)^2 \rightarrow$$
$$\hat{F} = a^2|n_0\rangle\langle n_0| + 2(n_0+1)ab|n_0+1\rangle\langle n_0+1| + (n_0+1)(n_0+2)b^2|n_0+2\rangle\langle n_0+2| \tag{3.7}$$

Here $f(z^\star, z)$ is nonnegative while $\hat{F}$ is indefinite if $ab < 0$. We then have the result :

$$\hat{\rho} \quad \text{classical or semiclassical} \Leftrightarrow P(I) \geq 0 \Rightarrow$$
$$a^2 p(n_0) + 2(n_0+1)abp(n_0+1) + (n_0+1)(n_0+2)b^2 p(n_0+2)$$
$$= \frac{1}{n_0!} \int_0^\infty dIP(I)e^{-I} I^{n_0}(a + bI)^2 \geq 0 \tag{3.8}$$

So again, violation of any of these "local" inequalities in $p(n)$ implies that $\hat{\rho}$ is strongly nonclassical.

A physically illuminating example of the distinction between classical and semiclassical $\hat{\rho}$, and passage from one to the other, is provided by the case of the Kerr medium. The argument is intricate and rests on two well known results. The first is Hudson's theorem [14] : if a (purestate) wavefunction $\psi_0(q)$ has a nonnegative Wigner function $W_0(q,p)$, then $\psi_0(q)$ is Gaussian and conversely; in that case $W_0(q,p)$ is also Gaussian. The second result is the general connection between $\phi(z)$ and $W(q,p)$ for any $\hat{\rho}$:

$$W(q,p) = 2\int \frac{d^2z'}{\pi} e^{-2|z-z'|^2}\phi(z'), \quad z = \frac{1}{\sqrt{2}}(q + ip). \tag{3.9}$$

This means that for classical $\hat{\rho}$ with $\phi(z) \geq 0$, $W(q,p) \geq 0$ as well. Now imagine a single mode radiation field in an initial coherent state $|z_0\rangle$ with $\phi_0(z) = \pi\delta^{(2)}(z - z_0)$, incident upon a Kerr medium [15]. This initial state is pure, classical,and has a Gaussian wave function $\psi_0(q)$. The Kerr medium Hamiltonian is of the form

$$H_{\text{Kerr}} = \alpha\, \hat{a}^\dagger \hat{a} + \beta(\hat{a}^\dagger \hat{a})^2 \tag{3.10}$$

Clearly the number states $|n\rangle$ are eigenstates of this Hamiltonian. Therefore the Poissonian photon number distribution

$$p(n) = e^{-I_0} I_0^n/n!, \quad I_0 = z_0^\star z_0, \tag{3.11}$$

of the input state $|z_0\rangle$ is preserved under passage through the Kerr medium. Likewise the funtion $P(I) = \delta(I - I_0)$ is left unaltered. Therefore the output state $|\psi\rangle$, which of course is pure, is either classical or semiclassical. However the form of H_{Kerr} shows that the output wavefunction is non Gaussian. Therefore by Hudson's theorem the corresponding $W(q,p)$ must become negative somewhere. Therefore by eq. (3.9) the output $\phi(z)$ cannot be nonnegative. Thus passage through the Kerr medium converts an incident coherent state, which is classical, into a final state which is semiclassical.

B. The Two-mode case

Now we sketch the extension of these ideas to the two-mode case. Here the operator commutation relations, number and coherent states, and the diagonal representation for $\hat{\rho}$, are as follows:

$$
\begin{aligned}
&[\hat{a}_r,\hat{a}_s^\dagger]=\delta_{rs},\ [\hat{a}_r,\hat{a}_s]=[\hat{a}_r^\dagger,\hat{a}_s^\dagger]=0,\quad r,\ s=1,\ 2\ ;\\
&|n_1,n_2\rangle=(n_1!n_2!)^{-1/2}\left(\hat{a}_1^\dagger\right)^{n_1}\left(\hat{a}_2^\dagger\right)^{n_2}|0,\ 0\rangle\ ,\\
&\left(\hat{a}_1^\dagger\hat{a}_1\text{ or }\hat{a}_2^\dagger\hat{a}_2\right)|n_1,n_2\rangle=(n_1\text{ or }n_2)\,|n_1,n_2\rangle\ ;\\
&|\underline{z}\rangle=|z_1,z_2\rangle=\exp\left(-\frac{1}{2}\underline{z}^\dagger\underline{z}\right)\sum_{n_1,n_2=0}^{\infty}\frac{z_1^{n_1}z_2^{n_2}}{(n_1!n_2!)^{1/2}}|n_1,n_2\rangle\ ;\\
&\hat{\rho}=\int d\mu(\underline{z})\ \phi(\underline{z})|\underline{z}\rangle\langle\underline{z}|\ ,\\
&d\mu(\underline{z})=d^2z_1\ d^2z_2/\pi^2\cdot
\end{aligned}
\tag{3.12}
$$

It is convenient at this point to go when necessary beyond purely real classical functions $f(\underline{z})$ in applying the normal ordering rule to obtain corresponding operators. From the general number states matrix elements of $\hat{\rho}$ we read off some operator correspondences generalising eqns. (3.1) (3.2):

$$
\begin{aligned}
\langle n_3,n_4|\hat{\rho}|n_1,n_2\rangle &= \mathrm{Tr}(\hat{\rho}|n_1,n_2\rangle\langle n_3,n_4|)\\
&= \int d\mu(\underline{z})\,\phi(\underline{z})\ e^{-\underline{z}^\dagger\underline{z}}\frac{z_1^{*n_1}z_2^{*n_2}z_1^{n_3}z_2^{n_4}}{\sqrt{n_1!n_2!n_3!n_4!}}\Longrightarrow\\
e^{-\underline{z}^\dagger\underline{z}}\frac{z_1^{*n_1}z_2^{*n_2}z_1^{n_3}z_2^{n_4}}{\sqrt{n_1!n_2!n_3!n_4!}}&\longrightarrow|n_1,n_2\rangle\langle n_3,n_4|\cdot
\end{aligned}
\tag{3.13}
$$

For one mode the phase transformations form the group $U(1)$. For two modes this generalises to the group $U(2)$ of (passive) transformations mixing the two orthonormal single photon modes. At the operator level this means that the annihilation operators $\hat{a}_r$ experience a general $U(2)$ matrix transformation conserving total photon number. Moreover these transformations are unitarily implemented on the two-mode Hilbert space [16]:

$$
\begin{aligned}
u=(u_{rs})\ \epsilon\ U(2)\ &:\\
\mathcal{U}(u)\ \hat{a}_r\ \mathcal{U}(u)^{-1}&=u_{sr}\ \hat{a}_s\ ,\\
\mathcal{U}(u)\ \hat{a}_r^\dagger\ \mathcal{U}(u)^{-1}&=u_{sr}^*\ \hat{a}_s^\dagger\ ,\\
\mathcal{U}(u)\,\mathcal{U}(u)^\dagger&=1\ ,\\
\mathcal{U}(u')\,\mathcal{U}(u)&=\mathcal{U}(u'u)\cdot
\end{aligned}
\tag{3.14}
$$

For later use we give here the actions of these unitary operators $\mathcal{U}(u)$ on monomials formed out of $\hat{a}_r^\dagger$ and $\hat{a}_r$, on the number states and on coherent states:

$$
\begin{aligned}
&u=e^{i\alpha}a\ \epsilon\ U(2),\ a\ \epsilon\ SU(2)\quad:\\
&\mathcal{U}(u)\frac{\hat{a}_1^{\dagger j+m}\hat{a}_2^{\dagger j-m}}{\sqrt{(j+m)!(j-m)!}}\mathcal{U}(u)^{-1}=e^{-2i\alpha j}\sum_{m'}D^{(j)}_{m'm}(a)\frac{\hat{a}_1^{\dagger j+m'}\hat{a}_2^{\dagger j-m'}}{\sqrt{(j+m')!(j-m')!}}\ ,\\
&\mathcal{U}(u)\frac{\hat{a}_1^{j+m}\hat{a}_2^{j-m}}{\sqrt{(j+m)!(j-m)!}}\mathcal{U}(u)^{-1}=e^{2i\alpha j}\sum_{m'}D^{(j)}_{m'm}(a)^*\frac{\hat{a}_1^{j+m'}\hat{a}_2^{j-m'}}{\sqrt{(j+m')!(j-m')!}}\ ,\\
&\mathcal{U}(u)|j+m,j-m\rangle=e^{-2i\alpha j}\sum_{m'}D^{(j)}_{m'm}(a)|j+m',j-m'\rangle\ ,\\
&j=0,\ 1/2,\ 1,\ldots,\\
&m,\ m'=j,\ j-1,\ \ldots,\ -j\ ;\\
&\mathcal{U}(u)|\underline{z}\rangle=|u^*\ \underline{z}\rangle\ \cdot
\end{aligned}
\tag{3.15}
$$

We have chosen the exponents of $\hat{a}$'s and $\hat{a}^\dagger$'s, the numerical factors, and the number operator eigenvalues, in such a way that the results can be expressed neatly using the $SU(2)$ representation matrices in various unitary irreducible $SU(2)$ representations, namely the $\mathcal{D}$-functions of quantum angular momentum theory [17].

To motivate the existence of several layers of classicality, we now generalise the single mode $U(1)$-invariant real factorial moments γ_n of eqn. (3.6) to two-mode quantities which conserve total photon number and also transform in a closed and covariant manner under $SU(2)$. For this purpose, keeping in mind eqns. (3.15), it is convenient to start with the (in general complex) classical monomials

$$
\begin{aligned}
f^j_{m_1 m_2}\left(\underline{z}^\dagger, \underline{z}\right) &= N_{jm_1m_2} z_1^{*j+m_1} z_2^{*j-m_1} z_1^{j+m_2} z_2^{j-m_2} \ , \\
N_{jm_1m_2} &= \left[(j+m_1)!\,(j-m_1)!\,(j+m_2)!\,(j-m_2)!\,\right]^{-1/2} \ , \\
j &= 0,\ 1/2,\ 1,\ \ldots, \\
m_1,\ m_2 &= j,\ j-1,\ \ldots,\ -j \cdot
\end{aligned}
\tag{3.16}
$$

The total power of $\underline{z}$ is equal to that of $\underline{z}^\dagger$, hence these are $U(1)$-invariant. The corresponding operators and their $SU(2)$ transformation laws are:

$$
\begin{aligned}
f^j_{m_1m_2}\left(\underline{z}^\dagger, \underline{z}\right) \rightarrow \widehat{F}^j_{m_1m_2} &= N_{jm_1m_2} \hat{a}_1^{\dagger j+m_1} \hat{a}_2^{\dagger j-m_1} \hat{a}_1^{j+m_2} \hat{a}_2^{j-m_2} \ ; \\
a \,\epsilon\, SU(2) : \mathcal{U}(a) \widehat{F}^j_{m_1m_2} \mathcal{U}(a)^{-1} &= \sum_{m_1', m_2'} D^{(j)}_{m_1'm_1}(a) D^{(j)}_{m_2'm_2}(a)^* \widehat{F}^j_{m_1'm_2'} \cdot
\end{aligned}
\tag{3.17}
$$

For a given two-mode state $\hat{\rho}$ we now generalise the factorial moments γ_n of eqn. (3.6) to the following three-index quantities:

$$
\begin{aligned}
\gamma^{(j)}_{m_2m_1} &= \mathrm{Tr}\left(\hat{\rho}\ \widehat{F}^j_{m_1m_2}\right) \\
&= N_{jm_1m_2}\ \mathrm{Tr}\left(\hat{\rho} \hat{a}_1^{\dagger j+m_1} \hat{a}_2^{\dagger j-m_1} \hat{a}_1^{j+m_2} \hat{a}_2^{j-m_2}\right) \cdot
\end{aligned}
\tag{3.18}
$$

Their $SU(2)$ transformation law is clearly

$$
\begin{aligned}
\hat{\rho}' &= \mathcal{U}(a)\ \hat{\rho}\ \mathcal{U}(a)^{-1} \ : \\
\gamma'^{(j)}_{m_2m_1} &= \sum_{m_1', m_2'} D^{(j)}_{m_1'm_1}\left(a^{-1}\right) D^{(j)}_{m_2'm_2}\left(a^{-1}\right)^* \gamma^{(j)}_{m_2'm_1'} \ , \\
\text{ie} \quad \gamma'^{(j)} &= D^{(j)}(a)\ \gamma^{(j)}\ D^{(j)}(a)^\dagger \cdot
\end{aligned}
\tag{3.19}
$$

In the last line for each fixed j the generalised moments $\gamma^{(j)}_{m_1m_2}$ have been regarded as a (hermitian) matrix of dimension $(2j+1)$.

On account of the fact that the total photon number is conserved in the definition of these moments, calculation of $\gamma^{(j)}_{m_1m_2}$ does not require complete knowledge of $\phi(\underline{z})$ but only of a partly angle averaged quantity $\mathcal{P}(I_1, I_2, \theta)$:

$$
\begin{aligned}
\gamma^{(j)}_{m_1m_2} &= N_{jm_1m_2} \int_0^\infty dI_1 \int_0^\infty dI_2 \int_0^{2\pi} \frac{d\theta}{2\pi} \\
&\qquad \mathcal{P}(I_1, I_2, \theta)(I_1 I_2)^j (I_1/I_2)^{1/2(m_2+m_1)} \cdot e^{i(m_1-m_2)\theta} \ , \\
\mathcal{P}(I_1, I_2, \theta) &= \int_0^{2\pi} \frac{d\theta_1}{2\pi}\ \phi\left(I_1^{1/2} e^{i\theta_1}, I_2^{1/2} e^{i(\theta_1+\theta)}\right) \cdot
\end{aligned}
\tag{3.20}
$$

It is clear that these moments $\gamma^{(j)}_{m_1m_2}$ involve more than just the photon number probabilities $p(n_1, n_2)$ which are just the "diagonal" case of the general matrix element in eqn. (3.13):

$$
\begin{aligned}
p(n_1, n_2) &= \langle n_1, n_2 | \hat{\rho} | n_1, n_2 \rangle \\
&= \int_0^\infty dI_1 \int_0^\infty dI_2\ P(I_1, I_2)\ e^{-I_1 - I_2} I_1^{n_1} I_2^{n_2} / n_1!\ n_2! \ , \\
P(I_1, I_2) &= \int_0^{2\pi} \frac{d\theta}{2\pi}\ \mathcal{P}(I_1, I_2, \theta) \cdot
\end{aligned}
\tag{3.21}
$$

This is the two-mode version of eqn. (3.3). The subset of "diagonal" moments $\gamma^{(j)}_{mm}$ are calculable in terms of $p(n_1, n_2)$ or $P(I_1, I_2)$:

$$
\begin{aligned}
\gamma^{(j)}_{mm} &= \int_0^\infty dI_1 \int_0^\infty dI_2\; P(I_1,I_2) I_1^{j+m} I_2^{j-m} \big/ (j+m)!\,(j-m)! \\
&= \sum_{n_1,n_2} p(n_1,n_2)\; n_1!\; n_2! \big/ (n_1-j-m)!(n_2-j+m)!(j+m)!(j-m)!
\end{aligned} \tag{3.22}
$$

However under a general $SU(2)$ mixing of the modes, the expressions $\gamma^{(j)}_{mm},\ p(n_1,n_2),\ P(I_1,I_2)$ do not transform in any neat way among themselves, and one is obliged to enlarge the set to include the more general $\gamma^{(j)}_{m_1 m_2}$ and $\mathcal{P}(I_1,I_2,\theta)$. (In particular, for these, the probabilities $p(n_1,n_2)$ are inadequate). When this is done we see the need to deal with both the quantities $\mathcal{P}(I_1,I_2,\theta),\ P(I_1,I_2)$ derived from $\phi(\underline{z})$ by a single or a double angular average. One can therefore distinguish four levels of classicality for two-mode states:

$$
\begin{aligned}
&\hat{\rho}\ \text{classical} \Leftrightarrow \phi(\underline{z}) \geq 0,\ (\text{hence } \mathcal{P}(I_1,I_2,\theta), P(I_1,I_2) \geq 0)\ ; \\
&\hat{\rho}\ \text{semiclassical } I \Leftrightarrow \mathcal{P}(I_1,I_2,\theta) \geq 0\ (\text{hence } P(I_1,I_2) \geq 0)\,,\ \text{but } \phi(\underline{z}) \ngeq 0\ ; \\
&\hat{\rho}\ \text{semiclassical } II \Leftrightarrow P(I_1,I_2) \geq 0 \text{ but } \mathcal{P}(I_1,I_2,\theta) \ngeq 0\ (\text{hence } \phi(z) \ngeq 0)\ ; \\
&\hat{\rho}\ \text{strongly nonclassical} \Leftrightarrow P(I_1,I_2) \ngeq 0 (\text{hence } \phi(\underline{z}),\ \mathcal{P}(I_1,I_2,\theta) \ngeq 0)\cdot
\end{aligned} \tag{3.23}
$$

These definitions can be cast in dual operator forms. For example, for semi classical -I states, we can say that for any classical real nonnegative overall $U(1)$ phase invariant $f\left(\underline{z}^\dagger, \underline{z}\right)$ the corresponding operator $\widehat{F}$ has a nonnegative expectation value, while this fails for some $f\left(\underline{z}^\dagger, \underline{z}\right)$ outside this class. In the semi-classical-II case, we have to further limit $f\left(\underline{z}^\dagger, \underline{z}\right)$ to be real nonnegative and invariant under independent $U(1)\times U(1)$ phase transformations in the two modes, to be sure that the expectation value of $\widehat{F}$ is nonnegative.

At this point we can see that these levels of classicality possess different covariance groups. Since under a general $U(2)$ transformation $\mathcal{U}(u), u \in U(2)$, the function $\phi(\underline{z})$ undergoes a point transformation, $\phi(\underline{z}) \to \phi'(\underline{z}) = \phi(u^T \underline{z})$, we see that the property of being classical is preserved by all $U(2)$ transformations. On the other hand, a general $U(2)$ transformation can cause transitions among the other three levels. The point transformation property is obtained for $\mathcal{P}(I_1,I_2,\theta)$ and $P(I_1,I_2)$ only under the diagonal $U(1)\times U(1)$ subgroup of $U(2)$; in fact $P(I_1,I_2)$ is invariant under $U(1)\times U(1)$, while $\mathcal{P}(I_1,I_2,\theta)$ suffers a shift in the angle argument θ. Thus one can see that each of the three properties of being semiclassical-I, semiclassical-II or strongly non-classical is only $U(1)\times U(1)$ invariant.

As the number of modes increases further, clearly the hierarchy of levels of classicality also increases.

Generalising inequalities of the form (3.6) for the diagonal quantities $\gamma^{(j)}_{mm}$, for $\hat{\rho}$ classical or semiclassical-I or semiclassical-II, is quite straightforward, since then we deal with the two modes separately. The more interesting, and quite nontrivial, problem is to look for matrix generalisations of eqn. (3.6), bringing in the entire matrices $\gamma^{(j)} = \left(\gamma^{(j)}_{m_1 m_2}\right)$, and looking for inequalities valid for states of the classical or semi-classical-I types. (Of course for any quantum state $\hat{\rho}$ we have the obvious property that $\gamma^{(j)}$, for each j, is hermitian positive semidefinite. This is the two-mode generalisation of $\gamma_n \geq 0$ in the one-mode case). However this is expected to involve use of the Racah-Wigner calculus for coupling of tensor operators, familiar from quantum angular momentum theory, inequalities for reduced matrix elements, etc [18].

In the next Section we undertake a study of the particular case $j=1$ which involves at most quartic expressions in $\hat{a}$'s and $\hat{a}^\dagger$'s. This is just what is involved in giving a complete account of the two-mode generalisation of the Mandel Q-parameter familiar in the single-mode case.

IV. GENERALISED PHOTON-NUMBER FLUCTUATION MATRIX FOR TWO-MODE FIELDS

For the one-mode case, with the single photon number operator $\widehat{N} = \hat{a}^\dagger \hat{a}$, we have some obvious inequalities valid in all quantum states, and others valid in semiclassical and classical states as defined in eqn. (3.4):

<u>Any state</u> :

$$
\begin{aligned}
\langle \widehat{N} \rangle &\equiv \mathrm{Tr}\ (\hat{\rho}\widehat{N}) \equiv \gamma_1 \geq 0\ ; && (4.1a)\\
\langle :\widehat{N}^2\ : \rangle &\equiv \mathrm{Tr}\ \left(\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2\right) \equiv \gamma_2 \geq 0\ ; && (4.1b)\\
\langle \widehat{N}^2 \rangle &\equiv \mathrm{Tr}\ \left(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\right) \equiv \gamma_2 + \gamma_1 \geq 0\ ; && (4.1c)\\
(\Delta N)^2 &\equiv \langle \widehat{N}^2 \rangle - \langle \widehat{N} \rangle^2 \equiv \langle (\widehat{N} - \langle \widehat{N} \rangle)^2 \rangle \equiv \gamma_2 + \gamma_1 - \gamma_1^2 \geq 0\ ; && (4.1d)
\end{aligned}
$$

semiclassical or classical state

$$\langle :\ \widehat{N}^2\ :\rangle - \langle\widehat{N}\rangle^2 \equiv (\Delta N)^2 - \langle\widehat{N}\rangle \equiv \gamma_2 - \gamma_1^2 \geq 0\ \cdot \tag{4.1e}$$

(Here the dots : : denote normal ordering). The Mandel Q-parameter is defined as [5]

$$Q \equiv \frac{(\Delta N)^2 - \langle\widehat{N}\rangle}{\langle\widehat{N}\rangle} \equiv \frac{\gamma_2 - \gamma_1^2}{\gamma_1}\ , \tag{4.2}$$

and it has the property of being nonnegative in classical and semiclassical states. Conversely if Q is negative, the state is definitely strongly nonclassical. The two cases $Q > 0$ and $Q < 0$ correspond respectively to super and subpoissonian photon number distributions.

The inequalities (4.1) are not all independent, as some imply others. We now give the generalisation of these in matrix form, to two-mode states.

We have to deal with four independent number-like operators $\widehat{N}_\mu, \mu = 0, 1, 2, 3$ which we define thus:

$$\begin{aligned}\widehat{N}_\mu &= \underline{\hat{a}}^\dagger \sigma_\mu \underline{\hat{a}} = (\sigma_\mu)_{rs} \hat{a}_r^\dagger \hat{a}_s\ ,\\ \hat{a}_r^\dagger \hat{a}_s &= \frac{1}{2}(\sigma_\mu)_{rs}\widehat{N}_\mu \cdot\end{aligned} \tag{4.3}$$

(Here σ_0 and σ_j are the unit and the Pauli matrices, and the sum on μ goes from 0 to 3). The expectation values of $\widehat{N}_\mu$ in a general state $\hat{\rho}$ are written as n_μ :

$$\begin{aligned}\langle\widehat{N}_\mu\rangle \equiv \mathrm{Tr}(\hat{\rho}\widehat{N}_\mu) &= \int d\mu(\underline{z})\phi(\underline{z})\underline{z}^\dagger\sigma_\mu\underline{z} = n_\mu\ ,\\ \langle\hat{a}_r^\dagger\hat{a}_s\rangle &\equiv \mathrm{Tr}\ \left(\hat{\rho}\hat{a}_r^\dagger\hat{a}_s\right) = \frac{1}{2}(\sigma_\mu)_{rs}\ n_\mu .\end{aligned} \tag{4.4}$$

Thus n_μ and the matrix $\gamma^{(1/2)} = \left(\gamma^{(1/2)}_{m_1 m_2}\right)$ are essentially the same. Since the 2×2 matrix $\left(\langle\hat{a}_r^\dagger\hat{a}_s\rangle\right)$ is always hermitian positive semidefinite, we see that the generalisation of inequality (4.1a) to the two-mode case is

$$n_0 - |\underline{n}| \geq 0\ \cdot \tag{4.5}$$

(All components of n_μ are real). It may be helpful to remark that the matrix $\gamma^{(1/2)}$ is analogous to the coherency matrix, and the quantities n_μ to the Stokes parameters, in polarisation optics [19].

Now we consider quadratic expressions in $\widehat{N}_\mu$ which are upto quartic in $\hat{a}_r^\dagger$ and $\hat{a}_r$ combined. To handle their normal ordering compactly, we first define certain quadratic expressions in $\hat{a}_r$, and their hermitian conjugates:

$$\begin{aligned}\widehat{A}_j = i\ \underline{\hat{a}}^T\sigma_2\ \sigma_j\underline{\hat{a}}\ , &\quad \widehat{A}_j^\dagger = -i\ \underline{\hat{a}}^\dagger\sigma_j\sigma_2\underline{\hat{a}}^*,\ j = 1, 2, 3\ ;\\ \hat{a}_r\hat{a}_s = -\frac{i}{2}\left(\sigma_j\sigma_2\right)_{rs}\widehat{A}_j\ , &\quad \hat{a}_r^\dagger\hat{a}_s^\dagger = \frac{i}{2}(\sigma_2\sigma_j)_{rs}\ \widehat{A}_j^\dagger \cdot\end{aligned} \tag{4.6}$$

Under the action of the unitary operators $\mathcal{U}(a)$ representing $SU(2)$, both $\widehat{A}_j$ and $\widehat{A}_j^\dagger$ transform as real three-dimensional Cartesian vectors. Now we can easily express the result of writing the product $\widehat{N}_\mu\widehat{N}_\nu$ as a leading normally ordered quartic term plus a remainder:

$$\begin{aligned}\widehat{N}_\mu\widehat{N}_\nu &= :\ \widehat{N}_\mu\widehat{N}_\nu\ :\ + (\ell_{\mu\nu\lambda} + i\ \epsilon_{0\mu\nu\lambda})\ \widehat{N}_\lambda\ ,\\ :\ \widehat{N}_\mu\widehat{N}_\nu\ : &= t_{\mu\nu jk}\widehat{A}_j^\dagger\widehat{A}_k\ ,\\ t_{\mu\nu jk} &= \frac{1}{2}\left(\delta_{\mu\nu}\delta_{jk} - \delta_{\mu j}\delta_{\nu k} - \delta_{\nu j}\delta_{\mu k} - i\ \delta_{\mu 0}\epsilon_{0\nu jk} - i\ \delta_{\nu 0}\epsilon_{0\mu jk}\right)\ ,\\ \ell_{\mu\nu\lambda} &= \delta_{\mu\nu}\delta_{\lambda 0} + \delta_{\mu 0}\delta_{\nu\lambda} + \delta_{\nu 0}\delta_{\mu\lambda} - 2\ \delta_{\mu 0}\ \delta_{\nu 0}\ \delta_{\lambda 0} \cdot\end{aligned} \tag{4.7}$$

Here $\epsilon_{\sigma\mu\nu\lambda}$ is the four-index Levi-Civita symbol with $\epsilon_{0123} = 1$. So the anti-commutators and commutators among $\widehat{N}_\mu$ and $\widehat{N}_\nu$ are:

$$\frac{1}{2}\left\{\widehat{N}_\mu, \widehat{N}_\nu\right\} = t_{\mu\nu jk}\widehat{A}_j^\dagger\widehat{A}_k + \ell_{\mu\nu\lambda}\widehat{N}_\lambda\ , \tag{4.8a}$$

$$[\widehat{N}_\mu, \widehat{N}_\nu] = 2\ i\ \epsilon_{0\mu\nu\lambda}\widehat{N}_\lambda \tag{4.8b}$$

(These latter are just the $U(2)$ Lie algebra relations). To accompany n_μ, in a general state we denote the expectation values of $\widehat{A}_j^\dagger \widehat{A}_k$ by q_{jk}:

$$\langle \widehat{A}_j^\dagger \widehat{A}_k \rangle \equiv \mathrm{Tr}\left(\hat{\rho}\widehat{A}_j^\dagger \widehat{A}_k\right) = q_{jk} \cdot \tag{4.9}$$

Clearly, (q_{jk}) is basically the matrix $\gamma^{(1)} = (\gamma^{(1)}_{m_1 m_2})$ and is always a 3×3 hermitian positive semidefinite matrix. This statement is the generalisation of inequality (4.1b). We can also generalise the inequalities (4.1c) (4.1d) by saying that for any quantum state the two matrices with elements given by

$$\langle \frac{1}{2}\{\widehat{N}_\mu, \widehat{N}_\nu\}\rangle = t_{\mu\nu jk} q_{jk} + \ell_{\mu\nu\lambda} n_\lambda \ , \tag{4.10a}$$

$$\begin{aligned} \Delta(\widehat{N}_\mu, \widehat{N}_\nu) &\equiv \frac{1}{2}\langle\{\widehat{N}_\mu - \langle\widehat{N}_\mu\rangle \ , \ \widehat{N}_\nu - \langle\widehat{N}_\nu\rangle\}\rangle \\ &= t_{\mu\nu jk} q_{jk} + \ell_{\mu\nu\lambda} n_\lambda - n_\mu \, n_\nu \end{aligned} \tag{4.10b}$$

are both 4×4 real symmetric positive semidefinite. As in the one-mode case the inequality obeyed by $\left(\Delta\left(\hat{N}_\mu, \hat{N}_\nu\right)\right)$ implies the one obeyed by the anticommutator matrix $\left(\langle\frac{1}{2}\left\{\hat{N}_\mu, \hat{N}_\nu\right\}\rangle\right)$

Now we search for matrix inequalities which are valid in two-mode classical or semi-classical-I states, but not necessarily in semi-classical-II or strongly nonclassical states. The key ingredient is the formula

$$\begin{aligned} &\langle : \ \frac{1}{2}\{\widehat{N}_\mu, \widehat{N}_\nu\} \ : \rangle - \langle\widehat{N}_\mu\rangle\langle\widehat{N}_\nu\rangle \\ &\quad = \int d\mu(\underline{z})\phi(\underline{z}) \left(\underline{z}^\dagger \sigma_\mu \underline{z} - n_\mu\right) \left(\underline{z}^\dagger \sigma_\nu \underline{z} - n_\nu\right) \\ &\quad = \int_0^\infty \int_0^\infty dI_1 \, dI_2 \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{P}(I_1, I_2, \theta) \left(\underline{\zeta}^\dagger \sigma_\mu \underline{\zeta} - n_\mu\right) \left(\underline{\zeta}^\dagger \sigma_\nu \underline{\zeta} - n_\nu\right) \ , \\ &\underline{\zeta} = \begin{pmatrix} I_1^{1/2} \\ I_2^{1/2} \quad e^{i\theta} \end{pmatrix} \cdot \end{aligned} \tag{4.11}$$

We can now draw the following conclusion:

Classical or Semi-classical-I state

$$\begin{aligned} &\left(\langle : \ \frac{1}{2}\{\widehat{N}_\mu, \widehat{N}_\nu\} \ : \rangle \ - \langle\widehat{N}_\mu\rangle\langle\widehat{N}_\nu\rangle\right) \\ &\quad \equiv \left(\Delta(\widehat{N}_\mu, \widehat{N}_\nu) \ - l_{\mu\nu\lambda}\langle\widehat{N}_\lambda\rangle\right) \ \geq 0 \cdot \end{aligned} \tag{4.12}$$

This is the intrinsic two-mode expression of super-poissonian statistics, and its violation (possible only in semi-classical-II or strongly nonclassical states) is an intrinsic signature of two-mode subpoissonian photon statistics. What makes this criterion nontrivial is the fact that for any n_μ obeying eq. (4.5) the 4×4 matrix $(l_{\mu\nu\lambda} n_\lambda)$ is real symmetric positive semi-definite.

It is interesting to pin down the way in which this matrix inequality (4.12) can go beyond a single-mode condition [20]. The most general normalised linear combination of the two mode-operators $\hat{a}_r$ is determined by a complex two-component unit vector $\underline{\alpha}$:

$$\begin{aligned} \hat{a}(\underline{\alpha}) &= \underline{\alpha}^\dagger \underline{\hat{a}} = \alpha_r^* \hat{a}_r \ , \\ \underline{\alpha}^\dagger \underline{\alpha} &= 1 \ ; \\ [\hat{a}(\underline{\alpha}) \ , \hat{a}(\underline{\alpha})^\dagger] &= 1 \cdot \end{aligned} \tag{4.13}$$

For every such choice of a single mode, the inequality (4.12) does imply the single-mode inequality (4.1e). We can see this quite simply as follows. Given $\underline{\alpha}$, we define the real four-component quantity $\xi_\mu(\underline{\alpha})$ by

$$\begin{aligned} \xi_\mu(\underline{\alpha}) &= \frac{1}{2}\underline{\alpha}^\dagger \sigma_\mu \underline{\alpha} \ : \\ \xi_0(\underline{\alpha}) &= |\underline{\xi}(\underline{\alpha})| = 1/2 \cdot \end{aligned} \tag{4.14}$$

Then, using the completeness of σ_μ expressed by

$$(\sigma_\mu)_{rs}\ (\sigma_\mu)_{tu} = 2\ \delta_{ru}\ \delta_{st}\ , \tag{4.15}$$

we have the consequences:

$$\begin{aligned} \xi_\mu(\underline{\alpha})\widehat{N}_\mu &= \hat{a}(\underline{\alpha})^\dagger \hat{a}(\underline{\alpha}) \equiv \widehat{N}(\underline{\alpha})\ , \\ \ell_{\mu\nu\lambda}\xi_\mu(\underline{\alpha})\xi_\nu(\underline{\alpha}) &= \xi_\lambda(\underline{\alpha})\cdot \end{aligned} \tag{4.16}$$

Indeed we easily verify that (leaving aside $\xi_\mu = 0$ identically)

$$\begin{aligned} l_{\mu\nu\lambda}\xi_\mu\xi_\nu = \xi_\lambda \Rightarrow\ & \text{either}\ \ \xi_0 = |\underline{\xi}| = 1/2 \\ & \Leftrightarrow \xi_\mu = \frac{1}{2}\underline{\alpha}^\dagger\sigma_\mu\underline{\alpha}\ , \\ & \quad \text{some}\ \ \underline{\alpha}\ \ \text{obeying}\ \ \underline{\alpha}^\dagger\underline{\alpha} = 1\,, \\ & \text{or}\quad \xi_0 = 1\ ,\ \underline{\xi} = 0\cdot \end{aligned} \tag{4.17}$$

Saturating the left hand side of (4.12) with the latter possibility, $\xi_\mu = \delta_{\mu 0}$, leads to the superpoissonian condition for the total photon number distribution. Saturating it with $\xi_\mu(\underline{\alpha})\xi_\nu(\underline{\alpha})$ we get as a consequence:

$$(\Delta\widehat{N}(\underline{\alpha}))^2 - \langle\widehat{N}(\underline{\alpha})\rangle \geq 0,\ \text{any}\ \underline{\alpha}\ \cdot \tag{4.18}$$

In this way the two-mode matrix "superpoissonian" condition (4.12) implies the scalar single mode super poissonian condition (4.1e) for every choice of normalised single mode with annihilation operator $\hat{a}(\underline{\alpha})$, as well as for the total photon number.

However, it is easy to see that *the information contained in the matrix inequality (4.12) is not exhausted by the collection of single mode inequalities (4.18) for all possible choices of (normalized)* $\underline{\alpha}$. Denoting the real symmetric matrix appearing on the lefthand side of (4.12) by $(A_{\mu\nu})$,

$$A_{\mu\nu} = \Delta(\widehat{N}_\mu, \widehat{N}_\nu) - \ell_{\mu\nu\lambda}\langle\widehat{N}_\lambda\rangle\ , \tag{4.19}$$

it is clear that

$$\begin{aligned} \xi_\mu\ A_{\mu\nu}\ \xi_\nu\ \geq 0\ \text{for all}\ \ \xi_\mu\ \ \text{obeying}\ \xi_0 = |\underline{\xi}| = 1/2 \\ \not\Rightarrow (A_{\mu\nu}) \geq 0 \end{aligned} \tag{4.20}$$

Indeed, the lefthand side here reads in detail:

$$\xi_\mu A_{\mu\nu}\xi_\nu = \frac{1}{4}A_{00} + A_{0j}\xi_j + \xi_j\xi_k A_{jk}\ ; \tag{4.21}$$

and the nonnegativity of this expression for all 3-vectors ξ_j with $|\underline{\xi}| = 1/2$ cannot exclude the possibility of the 3×3 matrix (A_{jk}) having some negative eigenvalues. Part of the information contained in the matrix condition (4.12) is thus irreducibly two-mode in character, a sample of this being:

$$(A_{\mu\nu}) \geq 0 \Rightarrow (A_{jk}) \geq 0\ \cdot \tag{4.22}$$

Admittedly to a limited extent, this situation is analogous to some well known properties of Wigner distributions. Thus the marginal distributions in a single variable obtained by integrating $W(q,p)$ with respect to p or with respect to q (or any real linear combination of q and p) are always nonnegative probability distributions, even though $W(q,p)$ is in general indefinite. So also here, it can well happen that for a certain state both A_{00} and $\xi_\mu(\underline{\alpha})A_{\mu\nu}\xi_\nu(\underline{\alpha})$ are nonnegative for all $\underline{\alpha}$, yet $(A_{\mu\nu})$ is indefinite.

There exists in the literature a well known inequality for two-mode fields, which when violated is a sign of nonclassicality [21]. It reads:

$$\begin{aligned} \langle\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1) - 2\hat{n}_1\hat{n}_2\rangle \geq 0, \\ \hat{n}_1 = \hat{a}_1^\dagger\hat{a}_1, \quad \hat{n}_2 = \hat{a}_2^\dagger\hat{a}_2, \end{aligned} \tag{4.23}$$

and evidently involves only diagonal elements of the matrix $(A_{\mu\nu})$. After rearranging the operators in normal ordered form one can see that

$$
\begin{aligned}
&\langle \hat{n}_1(\hat{n}_1-1)+\hat{n}_2(\hat{n}_2-1)-2\hat{n}_1\hat{n}_2\rangle\\
&=\langle \hat{a}_1^{\dagger 2}\hat{a}_1^2+\hat{a}_2^{\dagger 2}\hat{a}_2^2-2\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2\rangle\\
&=\frac{1}{2}(q_{11}+q_{22}-q_{33})\\
&=A_{33}+n_3^2\\
n_3 &=\langle \hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\rangle.
\end{aligned}
\tag{4.24}
$$

By our analysis, in any classical or semiclassical-I state the matix $(A_{\mu\nu})$ is positive semidefinite, so in particular A_{33} and even more so the expression $A_{33}+n_3^2$, are both nonnegative. Thus the inequality (4.23) is certainly a necessary condition for classical and semiclassical-I states. Conversely, if (4.23) is violated and $A_{33}+n_3^2$ is negative, then certainly A_{33} is negative as well, and the state is either semiclassical-II or strongly non-classical. However this condition is unnecessarily strong since it asks for A_{33} to be less than $-n_3^2$; as we have shown, even the weaker condition $A_{33}<0$ is sufficient to imply that the state is semiclassical-II or strongly nonclassical. Vice versa, our necessary condition $A_{33}\geq 0$ for a classical state or semiclassical-I state is stronger than the condition (4.23). In both directions, then, our conditions are sharper than the ones existing in the literature.

We conclude this Section by presenting a few examples bringing out the content of the matrix condition (4.12), in particular the possibility of its containing more information than all single-mode projections of it.

(a) Pair-coherent states:
These are simultaneous eigenstates of $\hat{a}_1\hat{a}_2$ and $\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2$ [22]:

$$
\begin{aligned}
\hat{a}_1\hat{a}_2|\zeta,q\rangle &=\zeta|\zeta,q\rangle, \quad \zeta\in \mathcal{C},\\
(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)|\zeta,q\rangle &=q|\zeta,q\rangle, \quad q=0,\pm1,\pm2\cdots
\end{aligned}
\tag{4.25}
$$

For $q\geq 0$ these states are given by

$$
|\zeta,q\rangle=N_q\sum_{n=0}^{\infty}\frac{\zeta^n}{[n!(n+q)!]^{1/2}}|n+q,n\rangle \tag{4.26}
$$

where N_q is a normalisation constant. It is known that in these states the second mode already shows sub-poissonian statistics [23]. Thus if we write the matrix (4.19) for these states as $(A_{\mu\nu}(\zeta,q))$, then even without having to nontrivially mix the modes we find:

$$
\begin{aligned}
&\underline{\alpha}=(0\,,\,1)^T, \quad \xi_\mu(\underline{\alpha})=\frac{1}{2}\,\underline{\alpha}^T\sigma_\mu\underline{\alpha}=(\ 1/2,\ \ 0,\ \ 0,\ \ -1/2\ ):\\
&\xi_\mu(\underline{\alpha})A_{\mu\nu}(\zeta,q)\xi_\nu(\underline{\alpha})<0.
\end{aligned}
\tag{4.27}
$$

The matrix $A(\zeta,q)$ is indefinite and the pair coherent states are therefore neither classical nor even semiclassical-I. Consistent with this, a direct numerical study of the least eigenvalue $l\,(A(\zeta,q))$ of$(A_{\mu\nu}(\zeta,q))$ for sample values of ζ and q, does show it to be negative.

(b) Two-mode squeezed vacuum: It has been shown elsewhere [24] that a two mode squeezing transformation is characterised by two independent intrinsic squeeze parameters a and b obeying $a\geq b\geq 0$. A representative of such a transformation is

$$
\mathcal{U}^{(0)}(a,b)=\exp\left[\frac{(a-b)}{4}(\hat{a}_1^{\dagger 2}-\hat{a}_1^2)\right]\exp\left[\frac{(a+b)}{4}(\hat{a}_2^{\dagger 2}-\hat{a}_2^2)\right]. \tag{4.28}
$$

The case $a=b$ essentially corresponds to the second mode alone being squeezed. For general $a\neq b$ we have genuine two-mode squeezing; while the (Caves-Shumaker) limit $b=0$ involves maximal entanglement of the two modes. We restrict our analysis to this limit in the sequel. Then the two-mode squeezed vacuum is characterised by the single parameter a and is

$$
\mathcal{U}^{(0)}(a,0)\,|0,0\rangle=\exp\left[\frac{a}{4}\left(\hat{a}_1^{\dagger 2}-\hat{a}_1^2+\hat{a}_2^{\dagger 2}-\hat{a}_2^2\right)\right]|0,0\rangle \tag{4.29}
$$

The matrix $(A_{\mu\nu}(a))$ can be explicitly computed and happens to be diagonal:

$$(A_{\mu\nu}(a)) = \text{Diag}\Big(\frac{1}{2}\,(-3+7\,\cosh(2\,a))\,\text{Sinh}(a)^2\,,\;2\,\cosh(2\,a)\,\text{Sinh}(a)^2\,,\\ -2\,\text{Sinh}(a)^2\,,\;2\,\cosh(2\,a)\,\text{Sinh}(a)^2\Big) \tag{4.30}$$

We see that for all $a > 0$ this is indefinite, since the third eigenvalue $A_{22}(a)$ is strictly negative. This is displayed in Figure 1a, for a in the range $0 < a < 1$. Thus for all $a > 0$ the state (4.29) is definitely neither classical nor semiclassical-I. On the other hand the leading diagonal element (eigenvalue) $A_{00}(a)$ dominates the others in the sense that for all choices of single mode the "expectation value" of $A(a)$ is nonnegative:

$$\xi_\mu(\underline{\alpha})A_{\mu\nu}(a)\xi_\nu(\underline{\alpha}) \geq 0, \quad \text{all} \quad \underline{\alpha} \tag{4.31}$$

Thus the squeezed vacuum (4.29) displays nonclassicality via subpoissonian statistics in an intrinsic or irreducible two-mode sense which never shows up at the one mode level for any choice of that mode. This is to be contrasted with the case of pair-coherent states discussed previously. At the same time the state (4.29) is also quadrature squeezed for all $a > 0$. Thus both these nonclassical features are present simultaneously.

(c) Two-mode Squeezed Thermal state: This is defined as follows (we again limit ourselves to the case $b = 0$):

$$\hat{\rho}(a,\beta) = \mathcal{U}^{(0)}(a,0)\hat{\rho}_0(\beta)\mathcal{U}^{(0)}(a,0)^{-1},\\ \hat{\rho}_0(\beta) = (1-e^{-\beta})^2\exp\Big[-\beta(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2)\Big], \tag{4.32}$$

At zero temperature $\beta \to \infty$ this goes over to the previous example (b). Once again the matrix $(A_{\mu\nu}(a,\beta))$ can be computed analytically and it turns out to be diagonal:

$$(A_{\mu\nu}(a,\beta)) = \left(-1+e^{\beta}\right)^2 \times \\ \text{Diag}\begin{pmatrix} \frac{1}{8}\left(13 - 14\,e^{\beta} + 13\,e^{2\,\beta} + 20\,(1-e^{2\,\beta})\,\cosh(2\,a) + 7\,(1+e^{\beta})^2\,\cosh(4\,a)\right), \\ \frac{1}{2}\left((1-e^{\beta})^2 + 2\,(1-e^{2\,\beta})\,\cosh(2\,a) + (1+e^{\beta})^2\,\cosh(4\,a)\right), \\ 1+e^{2\,\beta} + (1-e^{2\,\beta})\,\cosh(2\,a), \\ \frac{1}{2}\left((1-e^{\beta})^2 + 2\,(1-e^{2\,\beta})\,\cosh(2\,a) + (1+e^{\beta})^2\,\cosh(4\,a)\right) \end{pmatrix} \tag{4.33}$$

Now the third element $A_{22}(a,\beta)$ can become negative for low enough temperature $T = \beta^{-1}$ or high enough squeeze parameter a. The variation of the least eigenvalue $l\,(A(a,\beta))$ of $A(a,\beta)$ with respect to a in the range $0 \leq a \leq 1$, for various choices of β, is shown in Figures (1b,c,d). One can see that if the temperature is not too high, for sufficiently large a the element $A_{22}(a,\beta)$ becomes negative, indicating that the state has then become semi-classical-II or strongly nonclassical. (In comparison we recall that for quadrature squeezing to set in the parameter a must obey the inequality $a > \ln\,\text{Coth}(\frac{\beta}{2})$ ([24])) On the other hand as in example (b), the leading element $A_{00}(a,\beta)$ again dominates the others in the sense that

$$\xi_\mu(\underline{\alpha})\,A_{\mu\nu}(a,\beta)\,\xi_\nu(\underline{\alpha}) \geq 0, \quad \text{all}\,\underline{\alpha} \tag{4.34}$$

So once again, when $A_{22}(a,\beta) < 0$, the subpoissonian statistics is irreducibly two-mode in character. In Figures (1b,c,d) we have also indicated the value of the squeeze parameter a at which quadrature squeezing sets in. It is interesting to see that, for the states described here, at each temperature, the irreducible two-mode subpoissonian statistics occurs before squeezing. Therefore (limiting ourselves to low order moments of $\phi(\underline{z})$) there exists a range of squeeze parameter where the only visible nonclassicality is through such subpoissonian statistics.

The more general squeezed thermal state

$$\hat{\rho}(\beta,a,b) = \mathcal{U}^{(0)}(a,b)\hat{\rho}_0(\beta)\mathcal{U}^{(0)}(a,b)^{-1}, \tag{4.35}$$

has qualitatively similar properties. Detailed numerical studies presented elsewhere [20] have shown that these states also do not show subpoissonian statistics at the one-mode level. On the other hand, direct search for the least eigenvalue of $(A_{\mu\nu}(a,b,\beta))$ reveals that, for suitable values of β, a, b, this is negative.

We thus have several instructive examples of the situation indicated by eq. (4.20)

V. CONCLUDING REMARKS

We have presented a dual operator and expectation value based approach to the problem of distinguishing classical from non-classical states of quantised radiation, and thus brought out the significance of this classification in a new physically interesting manner. As the number of independent modes increases, this approach leads to finer and yet finer levels of nonclassical behaviour, in a steady progression. This has been followed up by a complete analysis of photon number fluctuations for two-mode fields, and a comprehensive concept of subpoissonian statistics for such fields going beyond what can be handled by techniques developed at the one-mode level.

In a previous paper we have set up the formalism needed to examine the possibility of two-mode fields showing subpoissonian statistics at the one-mode level in an invariant manner, by following the variation of the Mandel Q-parameter as one continuously varies the combination of the two independent modes into a single mode. One can see through the work of the present paper that that preparatory analysis is a necessary prerequisite to be able to pinpoint the aspects of subpoissonian statistics which are irreducibly two-mode in character. Examples (b) and (c) at the end of Section IV bring out this aspect vividly.

The inequality (4.23) has been strengthened by our approach to a sharper criterion to distinguish various situations:

$$\begin{aligned} &\text{Classical or Semiclassical-I} \Rightarrow A_{33} \geq 0;\\ &A_{33} < 0 \Rightarrow \text{Semiclassical-II or strongly nonclassical.} \end{aligned} \tag{5.1}$$

From eqns. (4.19, 4.24) we see that A_{33} has the following neat expression:

$$\begin{aligned} A_{33} &= (\Delta\hat{n_1})^2 - \langle\hat{n_1}\rangle + (\Delta\hat{n_2})^2 - \langle\hat{n_2}\rangle - 2\Delta\left(\hat{n}_1, \hat{n}_2\right)\\ &= \langle(\hat{n}_1 - \hat{n}_2)^2\rangle - (\langle\hat{n}_1 - \hat{n}_2\rangle)^2 - \langle\hat{n}_1 + \hat{n}_2\rangle \end{aligned} \tag{5.2}$$

It is thus expressible solely in terms of expectations and fluctuations of the original (unmixed) mode number operators $\hat{n}_1, \hat{n}_2$ and their functions. One can now see easily, again from equation (4.19), that the statements (5.1) are part of a wider set of statements involving only expectations of functions of $\hat{n}_1, \hat{n}_2$:

$$\begin{aligned} A_{00} &= \left(\Delta\hat{N}_0\right)^2 - \langle\hat{N}_0\rangle,\\ A_{03} &= A_{30} = \Delta\left(\hat{N}_0, \hat{N}_3\right) - \langle\hat{N}_3\rangle,\\ A_{33} &= \left(\Delta\hat{N}_3\right)^2 - \langle\hat{N}_0\rangle,\\ \hat{N}_0 &= \hat{n}_1 + \hat{n}_2 \quad , \quad \hat{N}_3 = \hat{n}_1 - \hat{n}_2; \end{aligned} \tag{5.3a}$$

$$\begin{aligned} &\text{Classical or semiclassical-I} \Rightarrow \begin{pmatrix} A_{00} & A_{03}\\ A_{30} & A_{33} \end{pmatrix} \geq 0;\\ &\begin{pmatrix} A_{00} & A_{03}\\ A_{30} & A_{33} \end{pmatrix} < 0 \quad \Rightarrow \text{Semiclassical-II or strongly nonclassical.} \end{aligned} \tag{5.3b}$$

All other inequalities involving matrix elements such as $A_{01}, A_{02}, A_{13} \cdots$ involve "phase sensitive" quantities going beyond $\hat{n_1}$ and $\hat{n_2}$.

Going back to the matrix $A = (A_{\mu\nu})$, we see that from its properties we cannot immediately distinguish between the classical and semiclassical-I situations, or between the semiclassical-II and strongly non-classical situations. In both the former, A is positive semidefinite; while if A is indefinite, one of the latter two must occur. It would be interesting, for pair coherent states or squeezed thermal states for instance, to be able to see, when A is indefinite, whether we have a semi-classical-II or a strongly non-classical state, and whether this depends on and varies with the parameters in the state.

Acknowledgements
Arvind thanks University Grants Commission, India for financial support.

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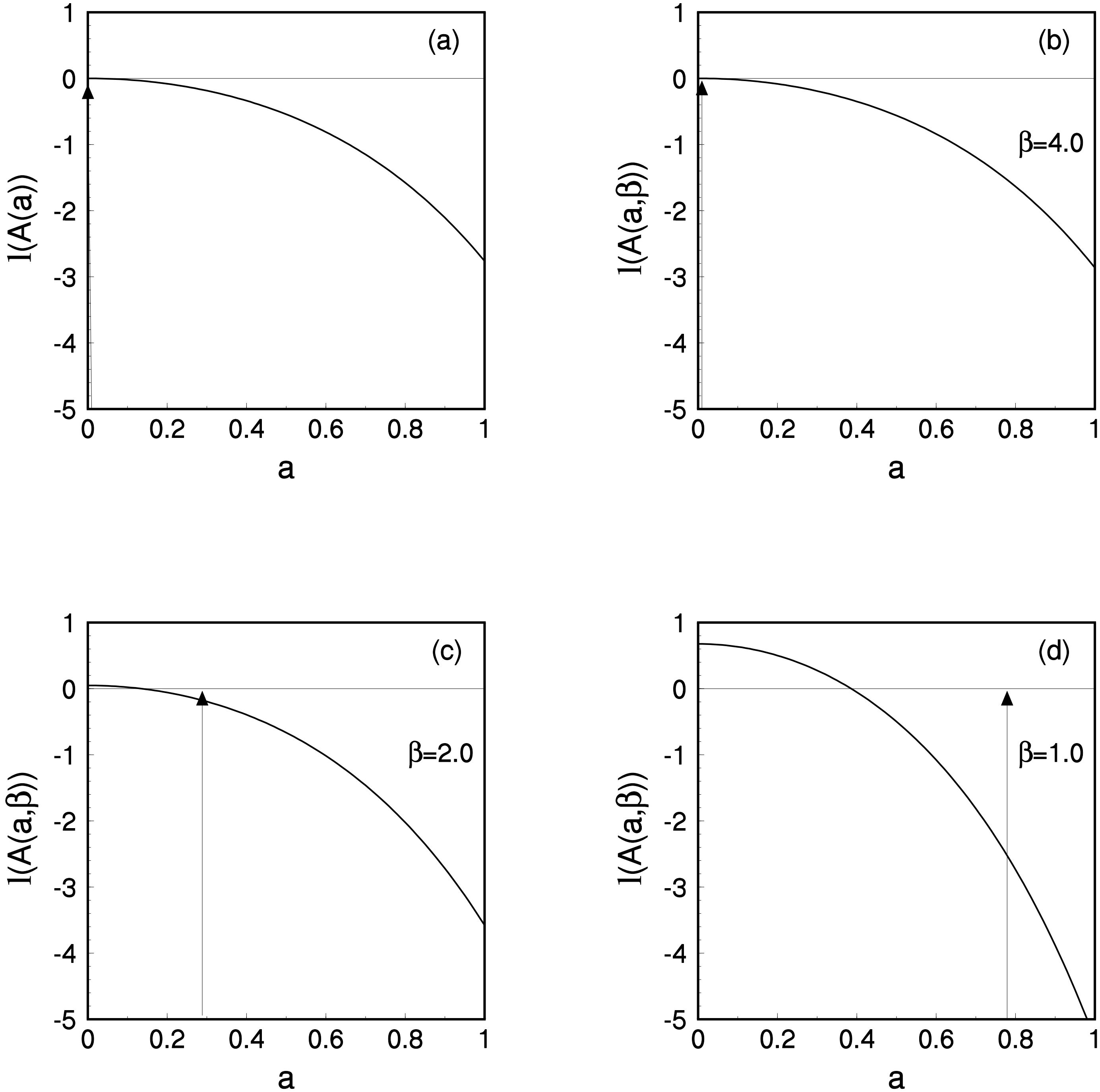
[1] M. C. Teich and B. E. A. Saleh in *Progress in Optics*, Vol. 26, ed. E. Wolf (North-Holland, Amsterdam, 1988); D. F. Walls, Nature **280**, 451 (1979).
[2] D. Stoler, Phys. Rev. D **1**, 3217 (1970); H. P. Yuen, Phys. Rev. A **13**, 2226 (1976); D. F. Walls, Nature **306**, 141 (1983)
[3] H. J. Kimble, M. Dagenais and L. Mandel, Phys. Rev. Letters **39** 691 (1977).
[4] R. Short and L. Mandel, Phys. Rev. Letters **51**, 384 (1983).
[5] L. Mandel, Optics Letters **4**, 205 (1979).
[6] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*, Benjamin, New York (1968); R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[7] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138** B274 (1965).
[8] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover New York, 1931), p.275.
[9] E. P. Wigner, Phys. Rev. A **40**, 749 (1932).
[10] C. K. Hong and L. Mandel, Phys. Rev. A **32**, 974 (1985).
[11] G. S. Agarwal and K. Tara, Phys. Rev. A **46** 485 (1992).
[12] C. L. Mehta (private communication).
[13] See J. R. Klauder and E. C. G. Sudarshan in ref. [6] above.
[14] R. L. Hudson, Rep. Math. Phys. **6**, 249 (1994).
[15] P. Tombesi and H. P. Yuen, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984) Vol. V p 751; G. J. Milburn, Phys. Rev. A **33**, 678 (1986).
[16] Arvind, B. Dutta, N. Mukunda and R. Simon,*The real symplectic groups in quantum mechanics and optics*, Pramana Jr. of Physics (1995) to appear.
[17] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics- Theory and Application* in *Encyclopedia of Mathematics and its Applications* Vol. 8 edited by G. C. Rota (Addison Wesley Publishing Co., Reading, Mass, USA 1981).
[18] L. C. Biedenharn and J. D. Louck, *The Racah-Wigner Algebra in Quantum Theory* in *Encyclopedia of Mathematics and its Applications* Vol. 9 edited by G. C. Rota (Addison Wesley Publishing Co., Reading, Mass, USA 1981).
[19] See for instance M. Born and E. Wolf *Principles of Optics*, (Pergamon, Elmsford NY, 1987).
[20] Arvind and N. Mukunda *Non-Classical Photon Statistics for Two-mode Optical Fields*, (IISc. preprint 1995).
[21] C. T. Lee, Phys. Rev. A **42**, 1608 (1990); R. Ghosh Current Science **67**, 520 (1994).
[22] D. Bhaumik, K. Bhaumik, and B. Dutta-Roy J. Phys. A **9**, 1507 (1976); G. S. Agarwal, Phys. Rev. Letters **57**, 827 (1986); G. S. Agarwal, JOSA B **5**, 1940 (1988).
[23] See G. S. Agarwal ref. [22] above.
[24] Arvind, B. Dutta, R. Simon and N. Mukunda, Phys. Rev. A **52**, 1609 (1995).

FIG. 1. Plots of least eigenvalue of the matrix $(A_{\mu\nu})$ as function of squeeze parameter a. Figure1(a) displays the least eigenvalue of $(A_{\mu\nu})$ for squeezed vacuum where as Figures1(b, c, and d) display the same for squeezed thermal states with inverse temperature β taking the values $4.0, 2.0$ and 1.0 respectively. In Figures1(b,c, and d) the arrow shows the setting in of quadrature squeezing.



(a)
(b)
β=4.0
(c)
β=2.0
(d)
β=1.0
l(A(a))
l(A(a,β))
a